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THE BOUNDARY OF CHAOS: AN INVESTIGATION INTO THE LENGTH RATIO DEPENDENT CHAOTIC DYNAMICS OF A PLANAR DOUBLE PENDULUM

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ABSTRACT. A planar double pendulum is defined by attaching two point masses together, with one of the point masses being connected to a pivot point. It is an interesting dynamic system because of its tendency to exhibit chaotic motion. Chaotic motion can be quantified using the Lyapunov exponent. If the Lyapunov exponent is positive, the system is considered chaotic. If the Lyapunov exponent is negative, instead of being chaotic, the system produces periodic motion. Extant research into the planar double pendulum indicates that as the length ratio of a pendulum increases, the Lyapunov exponent increases. Previous research has determined that at length ratio $1:1$ (comparing the length of the upper arm to the lower arm) the pendulum's motion is periodic, while if the length ratio is increased to $1:3$, the pendulum's motion is chaotic (Gupta et al., 2014). Building on Gupta et al. (2014)'s results, this research aimed to increase the precision of the measured length ratio representing the transitional point between periodic and chaotic motion. A computational simulation that provided a numerical solution to the Euler-Lagrange equations of the pendulum was used to determine the Lyapunov exponent for differing length ratios. The results demonstrated that the transitional length ratio lies between $1:2.34375$ and $1:2.375$, an increase in precision by a factor of 64 compared to the current bound established by extant research.

1 Literature Review

1.1 The Planar Double Pendulum System

A planar double pendulum is defined by attaching two point masses with a rigid, weightless rod, with the top point mass connected to a pivot point with a second rigid, weightless rod as seen in Figure 1 (Levien and Tan, 1993). The length ratio of a pendulum is expressed as $L_1 : L_2$. A pendulum is a Hamiltonian system, meaning its gravitational potential energy and kinetic energy is constantly exchanged and conserved throughout its motion (Biglari and Jami, 2016). Most importantly, the system has tendencies to produce chaotic motion (Richter and Scholz, 1984; Safitri et al., 2020).

1.2 The Phase Space

The phase space is an important mathematical tool that is used when describing a system's motion. In this research, the computational simulation was defined within a phase space coordinate set. A system's phase space is the graphical interpretation of the canonical coordinates that encode all possible physical states of the system (Nolte, 2018). As the system moves with time, a path is 'traced' within phase space, known as the phase space trajectory (Nolte, 2018). Every degree of freedom of the system is represented as a dimension of the multidimensional phase space (Nolte, 2018). In the case of a pendulum, these dimensions are $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$ (Levien and Tan, 1993), using the convention $\dot{f} = \frac{\partial}{\partial t}[f(t)]$.

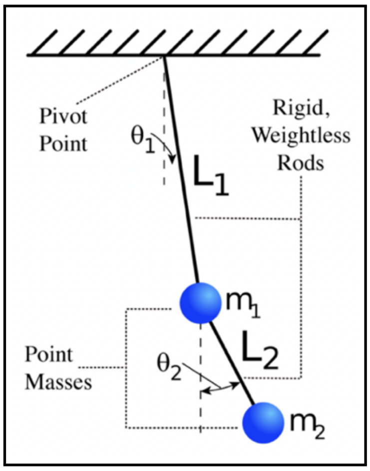


Figure 1
Diagrammatical representation of a pendulum system.
(Wikipedia, 2021).

1.3 Chaotic Motion and the Lyapunov Exponent

The Lyapunov exponent (λ in (1)) has proven to be the most useful quantification of chaos, and as such was used to quantify chaos in this research. A system is chaotic when $\lambda > 0$, and is periodic (motion repeated at set intervals) when $\lambda < 0$ (Wolf et al., 1985). Qualitatively, chaos is the physical phenomenon where a dynamic system is highly dependent on its initial conditions, and its motion is seemingly random (Gupta et al., 2014). λ is defined as the average exponential rate of divergence of infinitesimally close orbits in phase space (Wolf et al., 1985). Infinitesimally close orbits within phase space correspond to nearly identical physical states, hence an exponential divergence of these orbits implies a rapid loss of predictability of the system (Shivamoggi, 1997).

$$\lambda := \lim_{t\to\infty}\left[\lim_{\|\delta \mathrm{Z}_0\|\to 0}\left[\frac{1}{t}\ln\frac{\|\delta \mathrm{Z}_0(t)\|}{\|\delta \mathrm{Z}_0\|}\right]\right] \tag{1}$$

Formal definition of the Lyapunov exponent for a dynamic system.
(Wolf et al., 1985).

Danforth's algorithm[1] (Danforth, 2017) which determines λ (summarised by (2)) has been used to quantifythe chaos of a pendulum (Gupta et al., 2014; Levien and Tan, 1993). Despite the studies' use of Danforth's algorithm, none of them presented a complete and easily repeatable method for the algorithm. Therefore, this paper includes a repeatable summary of Danforth's algorithm for calculating λ of a pendulum in subsection 4.3, with the algorithm being generalised to any dynamic system in the Appendix (page 10).

$$\lambda_i(t) = \frac{1}{t}\sum_{n=1}^{t}\ln\left\|\mathbf{y}_n^i\right\| \tag{2}$$

Equation of the i-th largest Lyapunov exponent as a function of time.
(Danforth, 2017)

As per the details of Danforth's algorithm in the Appendix (page 10), it is suggested that in order for λ of a pendulum to be calculated, $\alpha = \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$ with the set $\left\{\varepsilon \overrightarrow{y_0^{\alpha}}\right\}$ being the set of vectors in (3).

$$\lim_{\varepsilon\to 0}\left\{\begin{bmatrix}\varepsilon\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\\varepsilon\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\\varepsilon\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\\varepsilon\end{bmatrix}\right\} \tag{3}$$

Column vector set for displacement vectors at the limit $\varepsilon \to 0$.

1.4 The Principle of Least Action

Rather than utilising Newton's second law of motion $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, the principle of least action was used to formulate the pendulum's simulation and Euler-Lagrange equations (Gray, 2009). Feynman et al. (1964)'s definition of the principle of least action is "the average kinetic energy less the average potential energy is as little as possible for the path of an object going from one point to another". The action functional S_i of a pendulum is:

$$\int_{t_0}^{t_1}\frac{1}{2}m\dot{\theta}_i^2 - mg\theta_i \, dt \tag{4}$$

Action functional between the time period t_1 and t_2 for a pendulum system.
(Gray, 2009).

This functional is relatively simple to compute numerically compared to the forces and acceleration of the masses. The actual path that is taken by the masses is that which minimises the action integral (Feynman et al., 1964; Gray, 2009). One consequence of this is the Euler-Lagrange equation seen in (5).

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_i} - \frac{\partial\mathcal{L}}{\partial\theta_i} = 0 \tag{5}$$

The Euler-Lagrange equations for a pendulum system.
(Deyst, 2003).

1.5 Extant Research

Extant research into the pendulum has predominantly been through the use of a computational simulation. This is due to the pendulum's sensitivity to its initial conditions, hence making it very challenging for a built model to undergo a valid testing method that can be reliably repeated.

Biglari and Jami (2016) provide information regarding the Kolmogorov–Arnold–Moser theorem. This theorem must be considered as it suggests that for certain initial conditions, the system may exhibit quasi-periodic motion, which is neither periodic nor chaotic. The theorem states that at low energies

[1] This algorithm has been adapted from Danforth's lecture Numerical Calculation of Lyapunov Exponents and his lecture notes *5.2 Numerical Calculation of Lyapunov Exponents*. However, the algorithm had been well-researched before this lecture was presented in 2017, and has been used to calculate λ previously.

(in the region of length ratio 1: 1 to 1: 3), the pendulum system's Euler-Lagrange equations may be integrable, meaning that if the phase space trajectory is subjected to a weak nonlinear perturbation, a portion of the invariant torus survives. This torus is the topological surface on which the phase space trajectory is bounded. Hence in this investigation, the motion of the pendulum near the transition point was investigated for possible quasiperiodicity, which can be seen if λ falls within the approximate range of 0 ± 0.05.

A study into the pendulum system by Stachowiak and Okada (2006) analysed the chaos of the system through the Lyapunov exponent. The study chose to investigate the dynamics of a pendulum in regard to its total energy E, and provided the knowledge that there is a clear boundary between periodic and chaotic motion at $E \approx 4.46$. This suggests that there are specific characteristics of a pendulum that makes it chaotic.

A study conducted Levien and Tan (1993) provides valuable information on λ as the initial angle increases. It was found that the system is chaotic if $\theta_1(0) > \frac{\pi}{3}$. This again showcases a specific characteristic of the pendulum system that makes it chaotic.

Gupta et al. (2014) explores the chaotic behaviour of a pendulum numerically. The simulation used by Gupta et al. (2014) was a MATLAB simulation, allowing them to measure how the mass and length ratios influenced the chaos of the system. It found that λ increases when the mass ratio is increased. It was also found that λ increases when the length ratio is increased, with the system being periodic at length ratio 1: 1 and chaotic at 1: 3. However, the researchers did not find a more precise length ratio at which the system transitions from periodic to chaotic motion. This research was designed as a follow on to Gupta et al. (2014)'s paper, with the goal being to increase the precision of the measured length ratio representing the transitional point between periodic and chaotic motion, referred to as the 'transitional length ratio' in this research.

2 Research Question

As the length ratio of a planar double pendulum increases (with initial small angle displacements), at what precise length ratio does the system transition from periodic to chaotic motion?

3 Hypothesis

That the length ratio at which a planer double pendulum system transitions from periodic to chaotic motion can be more precisely determined within the bound of 1: 1 and 1: 3 as established by extant research.

4 Methodology

4.1 Modelling the Dynamics of a Pendulum

The reasoning for this modelling was to determine the Euler-Lagrange equations of a pendulum system. These two equations (one for each mass) govern the dynamics of the masses, and formed the basis of the computational simulation. Some simplification steps have been omitted in the modelling for the sake of brevity, but all equations are accurate to the dynamics of the pendulum system.

The key initial conditions that must be defined for this system are the length of the pendulums' arm (L_i in metres), the point masses' mass (m_i in kilograms) and the angular displacement from the vertical of the two masses (θ_i in radians), where $i = 1, 2$ indexing the two point masses. These are labelled in Figure 1.

The Lagrangian $\mathcal{L}$ for the system is known and is given in (6).

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{2}(m_1 + m_2)(L_1)^2(\dot{\theta}_1)^2 \\
& + \frac{1}{2} m_2 (L_2)^2 (\dot{\theta}_2)^2 \\
& + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 + \theta_2) \\
& + g(m_1 + m_2) L_1 \cos\theta_1 \\
& + g m_2 L_2 \cos\theta_2
\end{aligned}
\tag{6}
$$

Using the Euler-Lagrange equation presented in (5), the equations of motion of the two masses can be obtained:

$$
\begin{aligned}
S_1 = L_1 \Big[& \ddot{\theta}_2 L_2 m_2 \cos(\theta_1 - \theta_2) \\
& + \left(\dot{\theta}_2\right)^2 L_2 m_2 \sin(\theta_1 - \theta_2) \\
& + (m_1 + m_2)\left(g \sin\theta_1 + L_1 \ddot{\theta}_1\right)\Big] = 0
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
S_2 = L_2 m_2 \Big[& -\left(\dot{\theta}_1\right)^2 L_1 \sin(\theta_1 - \theta_2) \\
& + \ddot{\theta}_1 L_1 \cos(\theta_1 - \theta_2) \\
& + \ddot{\theta}_2 L_2 + g(\sin\theta_2)\Big] = 0
\end{aligned} \tag{8}
$$

Due to their non-linear nature, there is no known method that solves the Euler-Lagrange equations analytically. However, they can be computed numerically using a computational program, one example being the *dsolve{<args>}* function in the Maplesoft computational simulator (Salisbury and Knight, 2002) which was used in this research. This method, however can provide some uncertainty within the Lyapunov exponent calculation as a numerical solution is not an exact solution to the differential equations.

4.2 Computing the Transitional Length Ratio Using the Bisection Method

In order to find precisely the transitional length ratio (where the pendulum transitions from periodic to chaotic motion), the bisection method was used. This method has not been used previously in research into a pendulum's dynamics, but is a common method for finding the zeros of polynomials. For this research, this method can be thought of as trying to find the length ratio that makes λ as close to 0 as possible ie. the length ratio's zero. During Test 1, the known transitional length ratio bound is between 1: 1 and 1: 3 as established from extant research (Gupta et al., 2014). The length ratio halfway between this bound (ie. 1: 2) will be tested and determined to be either chaotic or periodic. This will set a new bound for the transitional length ratio. The length ratio halfway between the new bound will then be tested, 'telescoping' the transitional length ratio to its precise value after repeating multiple times.

4.3 Steps Taken to Calculate The Lyapunov Exponent for Differing Length Ratios Using Danforth's Algorithm

1. Maplesoft computational program was generated to simulate the motion of a double pendulum system, using the Euler-Lagrange equations S_1 and S_2, the initial conditions in Table 1 and the *dsolve{<args>}* function.
2. Within the program, $\mathbf{v}_0$ was defined as the vector representing the initial conditions of the pendulum in the phase space of the pendulum system.
3. Within the program, $\varepsilon \mathbf{y}_n^\alpha$ where $\alpha = \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$ was defined as the basis displacement vectors of the four dimensions of the phase space at the limit as $\varepsilon \to 0$.
4. Using the simulating program, the five conditions $(\mathbf{v}_n, \varepsilon \mathbf{y}_n^\alpha)$ were iterated for a small time step (0.01 seconds), generating $\mathbf{v}_{n+1}$ and the four $\mathbf{k}_{n+1}^\alpha$.
5. The largest magnitude of the difference between the vectors $\mathbf{v}_{n+1}$ and $\mathbf{k}_{n+1}^\alpha$ was recorded, ie. $\left\|\mathbf{v}_{n+1} - \mathbf{k}_{n+1}^\alpha\right\| = \left\|\mathbf{y}_{n+1}^\alpha\right\|$.
6. The four $\mathbf{k}_{n+1}^\alpha$ were orthonormalised using GramSchmidt orthonormalisation, generating the four $\varepsilon \mathbf{y}_{n+1}^\alpha$.
7. Steps 4 to 6 were repeated for 150 seconds (ie. $n = 15000$) and $\lambda(t)$ was calculated utilising (2).
8. If λ was positive (ie. the system is chaotic), Steps 1 to 7 were repeated for the length ratio halfway between the tested ratio and the closest known ratio that produces periodic motion; if λ was negative (ie. the system is periodic), Steps 1 to 7 were repeated for the length ratio halfway between the tested ratio and the closest known ratio that produces chaotic motion.
9. Steps 1 to 8 were repeated six times, changing the length ratio each time as outlined in Step 8.

Table 1
Initial conditions used for the computational simulation.

Descritpion	Symbol	Value
Length of the top pendulum arm	L_1	1m
Length of the bottom pendulum arm	L_2	2m
Initial angular displacement of Mass 1 (in radians)	$\theta_1(0)$	0.2
Initial angular displacement of Mass 2 (in radians)	$\theta_2(0)$	0.2828
Initial angular velocity of Mass 1	$\dot{\theta}_1(0)$	0s^{-1}
Initial angular velocity of Mass 2	$\dot{\theta}_2(0)$	0s^{-1}
Mass of Mass 1	m_1	1kg
Mass of Mass 2	m_2	1kg
Local acceleration due to gravity	g	9.8m s^{-2}

5 Results

The Lyapunov exponent time series of each length ratio was generated within the Maplesoft simulation, producing the plots in Figure 2. These graphs show the value of λ on the x-axis plotted against time t on the y-axis for each of the seven tested length ratios. While for each length ratio λ initially fluctuated (even between positive and negative values) it then settled to a more consistent value which was observed and recorded to characterise the motion of the system. More specifically, it was determined whether the motion is periodic ($\lambda < 0$) and therefore if the length of the second arm was to be increased, or chaotic ($\lambda > 0$) and therefore the length of the second arm was to be decreased. The plots follow the chronological order of the length ratios which were tested, and demonstrate the process by which the transitional length ratio was 'telescoped' to a more precise measurement.

5.1 Visual Analysis

It can be observed that Figure 2a, 2c, 2e and 2f (on the following page) have a negative Lyapunov exponent. In contrast, Figure 2b and 2d show λ to be positive. At length ratio 1: 2.359375 (Figure 2g) it cannot be determined whether λ is positive or negative with certainty.

5.2 Numerical Analysis

The average value of λ between the time period of 40s to 150s was found within the Maplesoft computational program, using the integral in Equation (9). The time period of 40s to 150s was chosen as $\lambda(t)$ becomes relatively stable at $t = 40$, and the Maplesoft simulation could not compute $\lambda(t)$ for values > 150. The average value, the value at $t = 40$ and the value at $t = 150$ have been summarised in Table 1.

Table 2
Lyapunov exponent values for each length ratio evaluated through the Maplesoft computational simulation. Unlike Figure 2, these are arranged in order of increasing length ratio, depicting the change in λ occurring between 1: 2 and 1: 2.5.

	Length Ratio						
	1: 2	1: 2.25	1: 2.3135	1: 2.34375	1: 2.359375	1: 2.375	1: 2.5
	$\lambda(t)$						
$t = 40$	-3.03	-1.42	-0.26	-0.47	-0.14	-0.45	0.53
$t = 150$	-3.22	-1.31	-0.32	-0.28	-0.02	-0.56	0.49
Average	**-3.18**	**-1.40**	**-0.29**	**-0.24**	**0.05**	**0.37**	**0.51**
Interpretation	**Periodic**	**Periodic**	**Periodic**	**Periodic**	**Quasi-Periodic**	**Chaotic**	**Chaotic**

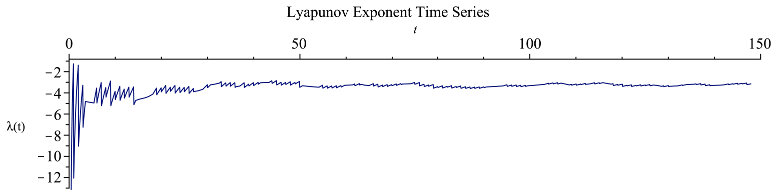
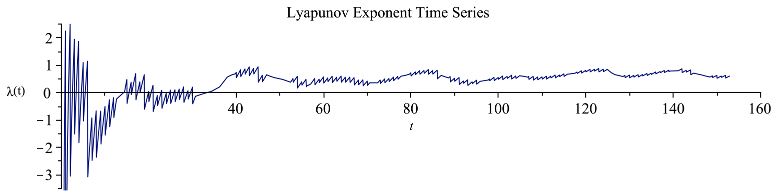
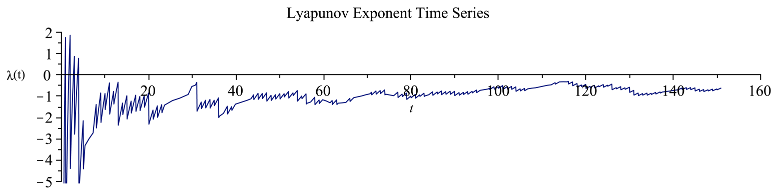
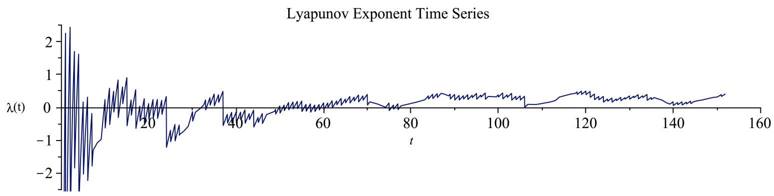
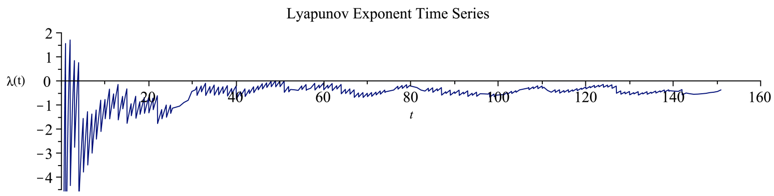
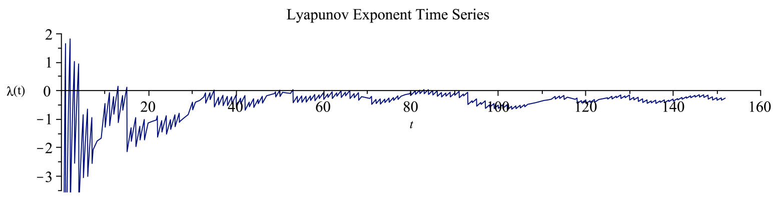
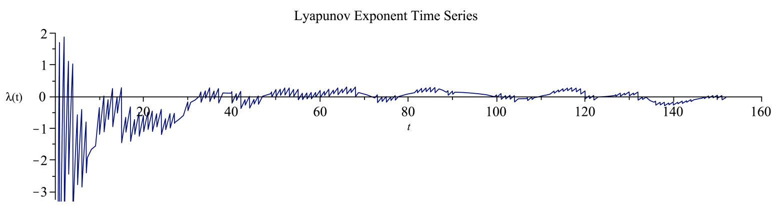
2a Test 1 1: 2	 *(a)* Lyapunov Time Series for Length Ratio 1 : 2	λ: Negative Interpretation: Periodic Action: Increase L_2
2b Test 2 1: 2.5	 *(b)* Lyapunov Time Series for Length Ratio 1 : 2.5	λ: Negative Interpretation: Periodic Action: Increase L_2
2c Test 3 1: 2.25	 *(c)* Lyapunov Time Series for Length Ratio 1 : 2.25	λ: Positive Interpretation: Chaotic Action: Decrease L_2
2d Test 4 1: 2.375	 *(d)* Lyapunov Time Series for Length Ratio 1 : 2.375	λ: Positive Interpretation: Chaotic Action: Decrease L_2
2e Test 5 1: 2.3125	 *(e)* Lyapunov Time Series for Length Ratio 1 : 2.3125	λ: Negative Interpretation: Periodic Action: Increase L_2
2f Test 6 1: 2.34375	 *(f)* Lyapunov Time Series for Length Ratio 1 : 2.34375	λ: Negative Interpretation: Periodic Action: Increase L_2
2g Test 7 1: 2.359375	 *(g)* Lyapunov Time Series for Length Ratio 1 : 2.359375	λ: Undetermined Interpretation: Quasi-Periodic Action: Stop Tests

Figure 2
Lyapunov exponent time series for initial length ratio of 1: 2 to length ratio 1: 2.359375.

(9) $$\int_{40}^{150} \frac{\lambda(t)\,dt}{110}$$

Expression to find the average value of $\lambda(t)$ between the time period 40s to 150s.

Using the data in Table 1, Figure 2 shows the relationship between the average Lyapunov exponent and the length ratio, expressed as the fraction $L_2 \div L_1$.

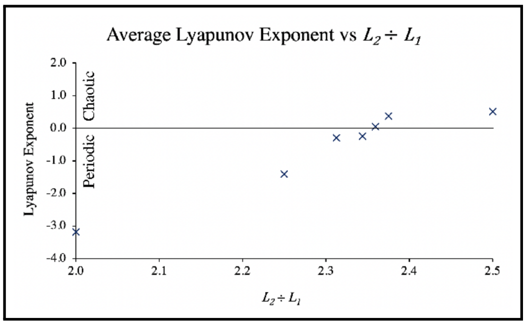


Figure 3
A plot of the average Lyapunov exponent as a function of the length ratio (expressed as the fraction $L_2 \div L_1$). It is important to note that λ and $L_2 \div L_1$ are both dimensionless quantities, hence no units are required.

6 Discussion

As per the literature review, if λ is < 0, the system is periodic, and if λ is > 0, the system is chaotic (Wolf et al., 1985). From the visual and numerical analysis of the Lyapunov exponent times series, between the length ratios $1\colon 2$ to $1\colon 2.34375, \lambda$ was negative and so the pendulum system was periodic. Furthermore, it was shown both visually and numerically that at length ratios between $1\colon 2.375$ and $1\colon 2.5$, the pendulum system was chaotic as λ was > 0. It can be inferred that the transitional length ratio lies between the length ratio of $1\colon 2.34375$ (the upper bound of periodic motion) and $1\colon 2.375$ (the lower bound of chaotic motion). This represents an improvement in precision of determining the transitional length ratio by a factor of 64 times in comparison to extant research (Gupta et al., 2014). As there was uncertainty in whether λ was positive or negative in Figure 2g, it was concluded that at the length ratio $1\colon 2.359375$, the pendulum produced quasiperiodic motion.

From the plot in Figure 3, it can be observed that there was a positive association between the Lyapunov exponent and the length ratio, however an exact linear correspondence between the two variables was not evident from the data. One reason this could arise is due to errors within the Lyapunov exponent calculation. However, this was not likely to be the cause of this nonlinear correspondence, as the computation simulation provided a numerical solution of the Euler-Lagrange equations that were accurate to 1 part per 10^6 (MapleSoft, 2012). The exact uncertainties of the Lyapunov exponent calculations are quite hard to derive, however they could be investigated in future research. Another explanation for this non-linear correspondence is that the two variables (length ratio and λ) are correlated through a third variable, which may cause a change in both λ and the length ratio.

One possible candidate of this third variable is the total energy E of the system, which is increased as the length ratio increases (Stachowiak and Okada, 2006). Furthermore, there is evidence that there is a clear boundary between periodic and chaotic motion at $E \approx 4.46$ (Stachowiak and Okada, 2006). E is generally expressed as $\sum_i [U_i + K_i]$ (OpenStax, 2016). As the system is a Hamiltonian (Assencio, 2014), E stays constant throughout the motion of the pendulum. Furthermore within this research, at $t = 0, K = 0$. Because $\frac{dE}{dL_2} = -gm_2 \cos\theta_2 = 9.41 > 0$, it is clear that when the length ratio is increased, the total energy of the system also increased.

It is proposed that the more energy the pendulum system has, the more likely it will be chaotic. This is because the phase space velocity will have a larger magnitude, and hence a slight perturbation to the phase space trajectory will have a larger proportional influence on the system. This may cause the phase space trajectory to diverge from its original path, ie. produce chaotic motion (Wolf et al., 1985).

Further research into the planar double pendulum might investigate the total energy of the system in two ways. The length ratio could be varied while ensuring that the total energy of the system stays constant throughout the tests. This can be achieved by changing a variety of variables $(m, \theta, \dot{\theta})$. If λ remains constant when the length ratio is changed and the total energy of the system is kept constant, it can be proposed that the length ratio is not the cause of the changing λ observed in this research. However, a relationship between E and λ would also need to be investigated. This can be done by keeping the length ratio constant and varying E. A proposed method would be to provide one mass with differing initial angular velocity, rather than the zero initial angular velocity that was used in this research.

It is most likely that the reason for a pendulum's chaotic motion is a combination of all the factors discussed above, however this is not yet clear from known research Chen (2008).

Finally, this research only focused on length ratios between 1: 1 and 1: 3. However at the limit as $L_2 \to \infty$, the planar double pendulum system can be thought of as a planar pendulum system (ie. only one mass on one rod), which is a periodic system (Parks, 2000). This suggests there is another transitional length ratio, where the pendulum transitions from chaotic to periodic motion. A conclusion that can be drawn from this is that there may be a finite range of length ratios of a double pendulum system that produce chaotic motion, which could be investigated in future research.

To summarise, it is proposed that the increase in length ratio may not solely be the cause of the increase in λ. Other qualities of the system, specifically the total energy E, should now be researched in order to determine if there are additional factors influencing the system's chaotic motion.

7 Conclusion

My research project explored the transitional length ratio between periodic and chaotic motion of a planar double pendulum system. Through the use of a computational simulation of a double pendulum, the chaos of the system was quantified and analysed through calculating the Lyapunov exponent (the accepted measure of chaotic motion). Previous research determined that the transitional length ratio lies between the bound of 1: 1 and 1: 3. The bisection method was used to increase the precision of this measurement, and it was determined that the transitional length ratio occurs between 1: 2.34375 and 1: 2.375, improving the precision of this measurement by a factor of 64 times. In doing so, my hypothesis was supported, that is, the precision of the transitional length ratio value can be increased from the bound established in current research.

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Appendix – Summary of Danforth (2017)'s Algorithm for Computing the Lyapunov Exponent

1. Define $\mathbf{v}_0$ as the vector representing the initial condition of the system in an i-th dimensioned phase space (ie. $\mathbf{v}_0 \in \mathbb{R}^i$).
2. Take a unit ball U_0 in $\mathbb{R}^i$ defined by the orthonormal basis set $\{\hat{\mathbf{w}}_0^1, \hat{\mathbf{w}}_0^2, \hat{\mathbf{w}}_0^3, \ldots \hat{\mathbf{w}}_0^i\}$[2] with centre $\mathbf{v}_0$.
3. Define the set $\{\epsilon\mathbf{y}_0^\alpha\}$ $(\alpha = 1, 2, 3, \ldots i)$ as the sum of $\mathbf{v}_0$ and $\hat{\mathbf{w}}_0^\alpha$.
4. Iterate the $i+1$ conditions $\{\mathbf{v}_0, \{\epsilon\mathbf{y}_0^\alpha\}\}$ for a small time step, generating $\mathbf{v}_1$ and the set $\{\mathbf{k}_1^\alpha\}$. This will transform U_0 into an ellipsoid with centred at $\mathbf{v}_1$ and the set $\{\mathbf{k}_1^\alpha\}$ lying on the ellipsoid's surface.
5. Record the of the difference between the vectors $\mathbf{v}_1$ and $\{\mathbf{k}_1^\alpha\}$, ie. $\|\mathbf{v}_1 - \mathbf{k}_1^\alpha\| = \|\mathbf{y}_1^\alpha\|$.
6. Orthogonalise the set $\{\mathbf{y}_1^\alpha\}$ using Gram-Schmidt orthonormalisation to generate the set $\{\hat{\mathbf{w}}_1^\alpha\}$. This transforms the ellipsoid into a unit ball.
7. Take a unit ball U_1 in $\mathbb{R}^i$ defined by the orthonormal basis set $\{\hat{\mathbf{w}}_1^\alpha\}$ with centre $\mathbf{v}_1$.
8. Define the set $\{\epsilon\mathbf{y}_1^\alpha\}$ as the sum of $\mathbf{v}_1$ and $\hat{\mathbf{w}}_1^\alpha$.
9. Repeat Steps 4 to 8 t times, and utilise Equation 2 to calculate $\lambda_i(t)$.

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[2]Subscript indexes iteration; superscript indexes dimension of decreasing expansion direction in the system's phase space vector space.